Communication for maths

Week 1: The mathematical communication of trigonometry

Introduction

- What these classes are not:
  - These classes are not maths tutorials;
  - We will not learn any new mathematics.

- What these classes are about:
  - learning how to correctly present mathematics;
  - communicating in mathematical English.

Introduction

- Mathematical solutions and proofs are not just a collection or sequence of steps.
- Mathematical solutions or proofs are supposed to tell a “story”.
- A rigorous solution includes a proper presentation of the steps, which includes proper use of mathematical English.
- It also helps you and the reader to make your math *readable.*

Introduction

- The following three slides give examples of mathematical presentation and communication from three different professional mathematics journal.

LEMMA 2.4. *Let $f\colon \mathbb{R}^d \to M$ be a K-quasiregular mapping, and let $\alpha \in \Omega^l(M)$ and $\beta \in \Omega^{d-l}(M)$ be closed forms. If B is a ball in $\mathbb{R}^d$ such that, in B, $f^*(\alpha \wedge \beta) = g dx^1 \wedge \cdots \wedge dx^d$ for a non-negative $g\colon B \to \mathbb{R}$, then*

$$\frac{1}{|\frac{1}{2}B|}\int_{\frac{1}{2}B} f^*(\alpha \wedge \beta) \leq C(d,K)\|\alpha\|_\infty \|\beta\|_\infty \left(\frac{1}{|B|}\int_B J_f^{d/(d+1)}\right)^{(d+1)/d},$$

where $C(d,K)$ depends only on d and K.

Proof. Let $\psi \in C_c^\infty(\mathbb{R}^d)$ be a non-negative function that is 1 on $\frac{1}{2}B$ and 0 on the complement of B. Note that we can choose ψ so that $|d\psi| \leq r^{-1}$, where r is the radius of B. By the non-negativity of $f^*(\alpha \wedge \beta)$,

$$\int_{\frac{1}{2}B} f^*(\alpha \wedge \beta) \leq \int_B \psi f^*(\alpha \wedge \beta).$$

On M, α is closed. By (2.3), $f^*\alpha = du$ on B. We can choose u so that u satisfies a Poincaré-Sobolev inequality. For a precise formulation of this, see [12, Cor. 4.2]. Integration by parts gives that

$$\left|\int_B \psi f^*\alpha \wedge f^*\beta\right| = \left|\int_B d\psi \wedge u \wedge f^*\beta\right|.$$

“A bound on the cohomology of quasiregularly elliptic manifolds”
Eden Prywes, *Annals of Mathematics* , Vol. 189, No. 3 (May 2019), pp. 863-883

THEOREM 2.1 (perturbation theorem). *For any $p \geq 2$, the functions u and $\hat{u}$ satisfy*

(2.2)
$$\|u-\hat{u}\|_{H_0^1(\Omega)} \leq \hat{r}^{-1}\|f-\hat{f}\|_{H^{-1}(\Omega)}+\hat{r}^{-1}\|\nabla u\|_{L_p(\Omega)}\|A-\hat{A}\|_{L_q(\Omega)}, \quad q := \frac{2p}{p-2} \in [2,\infty],$$

provided $\nabla u \in L_p(\Omega)$.

Proof. Let $\bar{u}$ be the solution to (1.1) with diffusion matrix $\hat{A}$ and right side f. Then, from the perturbation estimate (1.11), we have

$$\|\hat{u}-\bar{u}\|_{H_0^1(\Omega)} \leq \hat{r}^{-1}\|f-\hat{f}\|_{H^{-1}(\Omega)}. \tag{2.3}$$

We are therefore left with bounding $\|u-\bar{u}\|_{H_0^1(\Omega)}$. From the definition of u and $\bar{u}$, we have

$$\int_\Omega (A\nabla u)\cdot\nabla v = \int_\Omega (\hat{A}\nabla\bar{u})\cdot\nabla v$$

for all $v \in H_0^1(\Omega)$. This gives

$$\int_\Omega [\hat{A}\nabla(u-\bar{u})]\cdot\nabla v = \int_\Omega [(\hat{A}-A)\nabla u]\cdot\nabla v$$

for all $v \in H_0^1(\Omega)$. Taking $v = u-\bar{u}$, we obtain

$$\begin{aligned}\int_\Omega [\hat{A}\nabla(u-\bar{u})]\cdot\nabla(u-\bar{u}) &= \int_\Omega [(\hat{A}-A)\nabla u]\cdot\nabla(u-\bar{u})\\ &\leq \|(\hat{A}-A)\nabla u\|_{L_2(\Omega)}\|\nabla(u-\bar{u})\|_{L_2(\Omega)}.\end{aligned}$$

“Adaptive finite element methods for elliptic problems with discontinuous coefficients”
Andrea Bonito, Ronald A. Devore and Ricardo H. Nochetto,
SIAM Journal on Numerical Analysis, Vol. 51, No. 6 (2013), pp. 3106-3134

or

$$a_1(-w_4k + w_0k - 4w_3 + 4w_1) = -\lambda k, \\ -w_3k + w_1k - w_4 + w_0 = -\lambda. \tag{34}$$

Eliminating k, we obtain a quadratic equation for λ:

$$\lambda^2 + [(a_1 + 1)(w_0 - w_4)]\lambda + a_1[(w_0 - w_4)^2 - 4(w_1 - w_3)^2] = 0. \tag{35}$$

Using the values w_0, w_1, w_2, w_3, and w_4 given by (29), we obtain

$$(w_0 - w_4)^2 - 4(w_1 - w_3)^2 = 1, \tag{36}$$

and the equation for λ hence becomes

$$\lambda^2 + [(a_1 + 1)(w_0 - w_4)]\lambda + a_1 = 0. \tag{37}$$

Substituting the values a_1, w_0, and w_4 given by (22) and (29), we obtain

$$\lambda^2 - [e^{4\varphi+4\psi} + 1]\cosh(2\theta)\lambda + e^{4\varphi+4\psi} = 0. \tag{38}$$

We hence obtain two eigenvalues related to eigenvectors with the structure of eigenvectors (32), differing by k:

$$\lambda_{0,1} = \frac{(e^{4\varphi+4\psi} + 1)\cosh 2\theta}{2} \pm \sqrt{\frac{(e^{4\varphi+4\psi} + 1)^2 \cosh^2 2\theta}{4} - e^{4\varphi+4\psi}}. \tag{39}$$

“Eigenvalues of the transfer matrix of the three-dimensional Ising model in the particular case $n = m = 2$”, I. M. Ratner, *Theoretical and Mathematical Physics,* **199**(3): 909–921 (2019)

Introduction

- To achieve this degree of readability we need to learn three things:

  1. <u>Mathematical English</u> – Topic specific terminology: The words and phrasings of trig, algebra, differentiation, etc.;

  2. <u>General English vocabulary</u> – This is a restricted set of standard English words used throughout the solution/proof.

Introduction

- To achieve this degree of readability we need to learn three things:

  3. Correct presentation of solutions –
     - Correctly laid out;
     - Suitable terminology;
     - Suitable steps;
     - Cleanly presented.

Solutions and answers

Definition:

- The *solution* to a problem is
  a) the sequence of maths steps which lead logically to the correct answer, and which justify steps as and when necessary.
  b) The final answer to a problem: "The solution to a problem is the answer which solves the problem."

Solutions and answers

Definition:

- The *answer* to a problem is the last step/line in the solution described in a).
- … this also happens to be called the solution to the problem.
- So “solution” has two meanings:
  - The set of steps which illustrate how the problem was solved;
  - The final answer to the problem;

The presentation of solutions

Incorrect form of presentation:

If $\tan\theta = -\frac{3}{4}$, and θ is obtuse,

what is the value of $\sin\theta$?

Solution : $\sin\theta = +\frac{3}{5}$

This is not enough. This is just an answer. So what is the complete solution?

The presentation of solutions

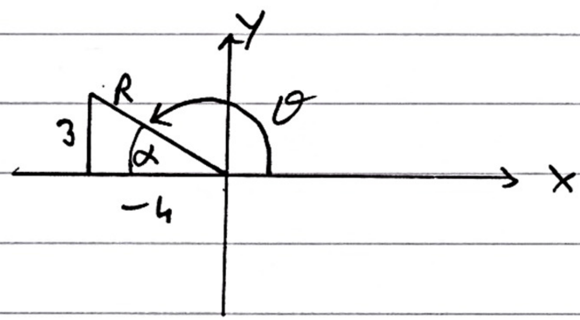
Correct form of presentation

If $\tan\theta = -\frac{3}{4}$, and θ is obtuse,

what is the value of $\sin\theta$?

Solution :

Consider the following diagram :



Since $R > 0$ we have

$R = \sqrt{9+16} = 5$ by Pythagoras' theorem.

Hence $\sin\theta = \frac{3}{5}$

The presentation of solutions

Example 1:

Consider the following question

$$\text{If } \tan\theta = -\frac{7}{24}, \text{ and } \theta \text{ is obtuse, find } \tan\frac{\theta}{2}$$

Let us now study the solution handed out to see what makes this a complete solution.

The presentation of solutions

In a solution we write in complete sentences.

- *Example*

Each step of a solution is a sentence:

Since $\cot A = 3$

we have $\dfrac{\cos A}{\sin A} = 3$.

The presentation of solutions

In a solution we do not write in columns.

- *Example*

  Consider the following presentation to a solution

Since $\cot A = 3$

we have $\dfrac{\cos A}{\sin A} = 3$.

$\therefore \cos A = 3 \sin A$ (*)

Since $\cos^2 A + \sin^2 A = 1$

we have by (*)

$9 \sin^2 A + \sin^2 A = 1$

$\therefore 10 \sin^2 A = 1$

The presentation of solutions

In a solution we do not write in columns.

- *Example*

  Reading from left to right we have

Since $\cot A = 3$ Since $\cos^2 A + \sin^2 A = 1$

The presentation of solutions

In a solution we do not write in columns.

- *Example*

Reading from left to right we have

we have $\frac{\cos A}{\sin A} = 3$.   we have by (*)

The presentation of solutions

In a solution we do not write in columns.

- *Example*

  Reading from left to right we have

$$9 \sin^2 A + \sin^2 A = 1$$

$$\therefore \cos A = 3 \sin A \quad (*) \qquad \therefore 10 \sin^2 A = 1$$

The presentation of solutions

In a solution we do not write in columns.

- *Example*

This does not make sense. How would you re-write the above solution to an appropriate form of presentation?

The presentation of solutions

In a solution we don't draw arrows (unless they are part of the maths of the solution)

Example

Since $\cot A = 3$

we have $\dfrac{\cos A}{\sin A} = 3$

$\therefore \quad \cos A = 3 \sin A$

By $\cos^2 A + \sin^2 A = 1$

$10 \sin^2 A = 1$

No arrows like these in solutions

The presentation of solutions

In a solution you <u>regularly</u> need to justify your steps.

<u>Example 1</u>

- "Today's date is 1st January"
- "A farmer who was rich decided to give his children some money. One day farmer asked to speak to his children."

The presentation of solutions

In a solution you <u>regularly</u> need to justify your steps.

<u>Example 2</u>

Since $\cot A = 3$, we have $\dfrac{\cos A}{\sin A} = 3$.

$\therefore \cos A = 3 \sin A$

Hence $10 \sin^2 A = 1$ ← **How so? Where does this come from?**

The presentation of solutions

In a solution you <u>regularly</u> need to justify.

<u>Example 3</u>

hence $(2\sin\theta + 1)(2\sin\theta - 3) = 0$

$\therefore$ $2\sin\theta + 1 = 0$ &/or $2\sin\theta - 3 = 0.$

So $\sin\theta = -\frac{1}{2}$

This is not a complete solution. Why?

The presentation of solutions

In a solution you <u>regularly</u> need to justify.

<u>Example 3</u>

hence $(2\sin\theta + 1)(2\sin\theta - 3) = 0$

$\therefore$ $2\sin\theta + 1 = 0$ & / or $2\sin\theta - 3 = 0$.

So $\sin\theta = -\frac{1}{2}$ & / or $\sin\theta = \frac{3}{2}$.

Since $\sin\theta \in [-1, 1]$, $\sin\theta = \frac{3}{2}$ is not valid.

Hence $\sin\theta = -\frac{1}{2}$.

OK

The presentation of solutions

Justification

Justification steps demonstrate your mathematical understanding of why a future step is what it is.

The presentation of solutions

In a solution use exact values where possible

- *Example*

.... hence $(2\sin\theta+1)(5\sin\theta-2)=0$.

$\therefore$ $2\sin\theta+1=0$ or $5\sin\theta-2=0$.

From which $\sin\theta=-\frac{1}{2}$ or $\sin\theta=\frac{2}{5}$.

Hence $\theta=-\frac{\pi}{6}$ or $\theta=0.41$ Rad.

The presentation of solutions

In a solution symbols should be use correctly

- *Example*

... hence $(2\sin\theta + 1)(5\sin\theta - 2) = 0$

$\longrightarrow$ $2\sin\theta + 1 = 0$ & / OR $5\sin\theta - 2 = 0$.

No. Why? Where is the mistake in the use of symbols?

The presentation of solutions

In a solution symbols should be use correctly

- *Example*

... hence $(2\sin\theta + 1)(5\sin\theta - 2) = 0$

$\Rightarrow \quad 2\sin\theta + 1 = 0$ oR/and $5\sin\theta - 2 = 0$

↑

Yes.

For our purpose the arrow symbol "→ " is used only in differentiation.

The presentation of solutions

Present final answers correctly

- *Example*

$$25 \sin(\theta + \alpha) = 25 \sin\left(\theta + \arctan \frac{7}{24}\right)$$

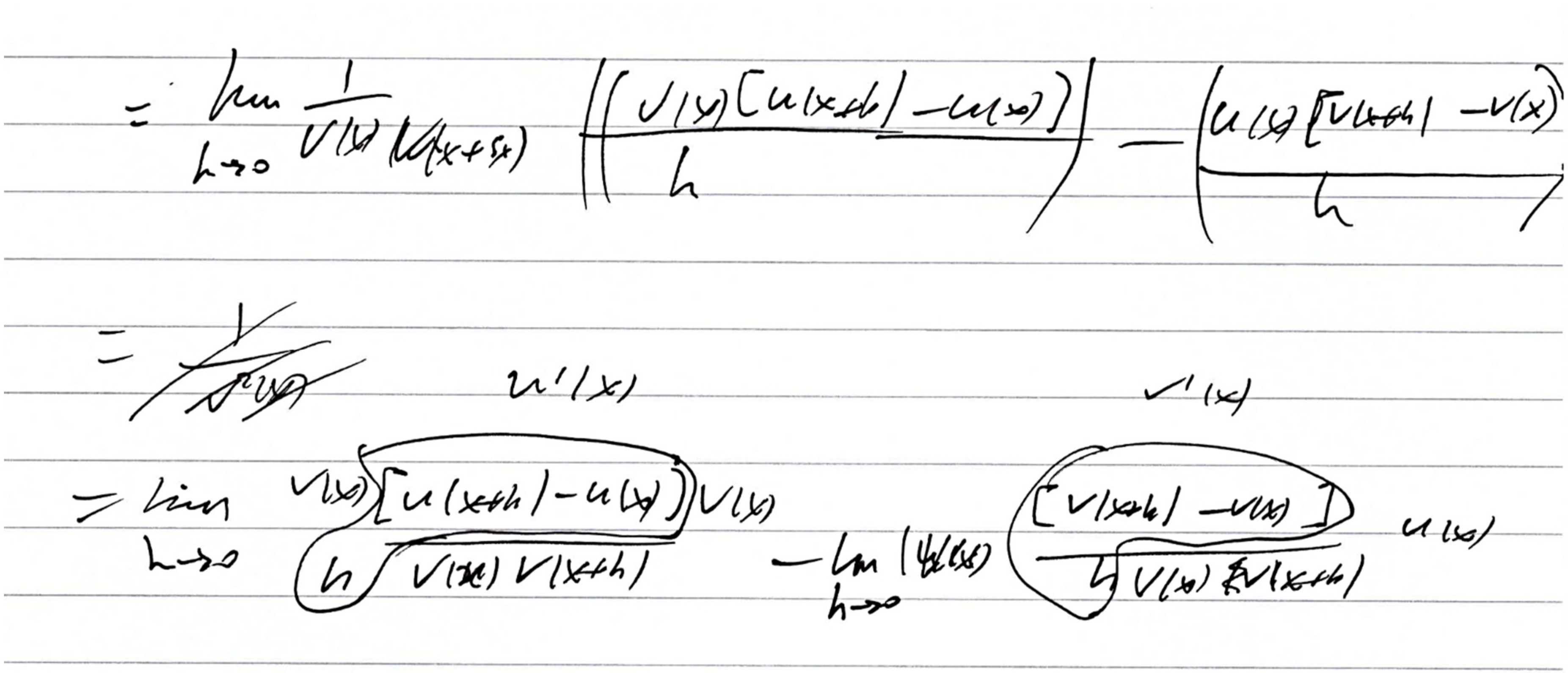
$$\arctan \frac{7}{24} \simeq 0.28 \simeq 16.26^\circ$$

$$\therefore \ \alpha \simeq 0.28 \,/\, 16.26^\circ$$

No. Why? What is a correct form of presentation?

The presentation of solutions

Make all of your work clean and readable



This is not readable

The presentation of solutions

Notation: The implication symbol

- *Example*

$$x = 2 \to x^2 = 4$$ No

$$x = 2 \Longrightarrow x^2 = 4$$ Yes

The presentation of solutions

Notation: The implication symbol

- *Example*

$$(a+b)^2 \Longrightarrow a^2 + 2ab + b^2$$ **No**

$$(a+b)^2 = a^2 + 2ab + b^2$$ **Yes**

Trigonometry problem

Example 2

Consider the following question

If cot A = 3, where $A \in [-\pi, 0]$, find cos A and sin A.

Let us now study the solution handed out to see what makes this a complete solution.

Trigonometry problem

Example 3

Consider the following question

Verify that

$$(1+\sin x)[1+\sin(-x)] = \cos^2 x$$

Let us now study the solution handed out to see what makes this a complete solution.

Trigonometry problem

Exercise 1

For the “solution” handed identify all cases of

- steps not written as complete sentences;
- writing in columns;
- the wrong use of symbols;
- lack of justification;
- exact values not being stated;
- messy presentation;

Trigonometry problem

Exercise 2

For the “solution” handed out find and complete the missing steps needed in order to make this a complete solution.

Trigonometry problem

Exercise 3

Write a complete solution to the following problem:

Find all solutions to $4 - 5\cos x = 2\sin^2 x$

in the interval $[0, 2\pi]$.

Final comment

As a lecturer my toughest initial task in turning enthusiastic students into able mathematicians is to force them (yes, force them) to write mathematics correctly. Their first submitted assessments tend to be incomprehensible collections of symbols, with no sentences or punctuation. 'What's the point of writing sentences?' they ask, 'I've got the correct answer. There it is – see, underlined – at the bottom of the page.' I can sympathise but in mathematics we have to get to the right answer in a rigorous way and we have to be able to show to others that our method is rigorous.

How to write mathematics, Kevin Houston, 2009, University of Leeds.

Final comment

A common response when I indicate a non-sensical statement in a student's work is 'But you are a lecturer, you know what I meant'. I have sympathy with this view too, but there are two problems with it.

(i) If the reader has to use their intelligence to work out what was intended, then the student is getting marks because of the reader's intelligence, not their own intelligence.

How to write mathematics, Kevin Houston, 2009, University of Leeds.

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Appendix

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The presentation of solutions

In a solution we do not write in columns.

- *Example*

  Consider the following presentation to a solution

ΔCBL	ΔCLA
$a^2 = (c+x)^2 + h^2$	$b^2 = h^2 + x^2$

If I read from left to right in the standard fashion, I read

$$\Delta CBL \ \Delta CLA \ a^2 = (c+x)^2 + h^2 \ b^2 = h^2 + x^2.$$

Trigonometry problem

Exercise 4

*(*Sine rule, p33, "how to think like a mathematician")*

The presentation of solutions

Notation

- <u>*Example 1*</u>

1. *(*Notation is another issue - I fairly often saw an angle described as 1.8(pi). We never write an angle with decimals and pi- either a pure decimal or a 'nice' multiple of (pi).*
2. *If you are asked to find angles between -(pi) and pi, then radian measure is wanted, not degrees. Degrees are not used in calculus. Unless we actually ask for an answer in degrees, you should always assume that radian measure is required. *)*